

PRODUCTS OF BRAUER SEVERI SURFACES

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ABSTRACT. Let $\{P_i\}_{1 \leq i \leq r}$ and $\{Q_i\}_{1 \leq i \leq r}$ be two collections of Brauer Severi surfaces (resp. conics) over a field k . We show that the subgroup generated by P_i 's in $Br(k)$ is the same as the subgroup generated by Q_i 's $\iff \Pi P_i$ is birational to ΠQ_i . Moreover in this case ΠP_i and ΠQ_i represent the same class in $M(k)$, the Grothendieck ring of k -varieties. The converse holds if $char(k) = 0$. Some of the above implications also hold over a general noetherian base scheme.

1. INTRODUCTION

1.1 (Notation). Let S denote a noetherian base scheme. All products, unless otherwise mentioned, will be over S . The class of any Brauer Severi scheme P over S in $Br(S)$ (the Brauer group of S) will be denoted by P itself. For a collection of Brauer-Severi schemes $\{P_i\}_{i \in I}$ over S , the subgroup generated by the P_i 's in $Br(S)$ will be denoted by $\langle \{P_i\}_{i \in I} \rangle$. $M(S)$ will denote the Grothendieck ring of finite type S -schemes (see section(2)).

All schemes considered will be noetherian. By a closed subscheme we will always mean a reduced closed subscheme.

The main result of this paper is the following.

1.2. **Theorem.** *Let $\{P_i\}_{1 \leq i \leq r}$ and $\{Q_j\}_{1 \leq j \leq r}$ be two collections Brauer Severi surfaces (resp. conics) over S . Consider the following conditions.*

- (i) $\langle \{P_i\} \rangle = \langle \{Q_j\} \rangle$ in $Br(S)$.
- (ii) $[\Pi P_i] = [\Pi Q_j]$ in $M(S)$.
- (iii) ΠP_i and ΠQ_j are birational.

Then (i) \Rightarrow (ii). If S is reduced then (i) \Rightarrow (iii). If S is a separated regular scheme then (i) \iff (iii). If S is a separated regular scheme with characteristic zero generic points, then (i) (ii) and (iii) are equivalent.

This result has been inspired by [2] where relations between products of conics in the Grothendieck ring were studied for the first time. The above theorem was proved in [2] for conics in the case when $S = \text{Spec}(k)$ where k is a number field or function field of an algebraic surface over \mathbb{C} .

The proof presented here is by induction on r . Working over a general noetherian base scheme S instead of a field enables us to run the induction more smoothly.

Recall the following conjecture of Amitsur.

1.3. Conjecture ([1]). *Let k be a field and P and Q be n -dimensional Brauer-Severi varieties over k . Then P is birational to $Q \iff P$ and Q generate the same subgroup in $Br(k)$.*

This conjecture is still unknown in general, however the following special cases are known.

- (1) P is split by a cyclic extension (which is always true if k is a local or global field) (see [1]).
- (2) $index(P) < dim(P) + 1$ (see [4]).
- (3) $P = -Q$ in $Br(k)$ (this proves the conjecture for Brauer-Severi surfaces) (see [4]).
- (4) $P = 2Q$ in $Br(k)$ (see [5]).

1.4. Remark. In addition to Brauer-Severi surfaces and conics, the proof of (1.2) presented here also works for Brauer-Severi varieties of prime index if one assumes Amitsur's conjecture for this case.

2. PRELIMINARIES ON THE GROTHENDIECK RING

2.1. (Grothendieck Ring). Let S be any scheme. Let $M(S)$ denote the free abelian group generated on isomorphism classes of reduced finite type S -schemes modulo the relations

$$[X] = [U] + [Z]$$

where X is a reduced S scheme and $U \subset X$ is an open subset with complement Z (with reduced scheme structure). For any S scheme X , we will use the notations $[X]_S$ or just $[X]$ to denote the class of X^{red} in $M(S)$. For S schemes X, Y define

$$[X]_S \cdot [Y]_S = [(X \times_S Y)]_S$$

This makes $M(S)$ into a commutative and associative ring with $[S]$ being the identity in this ring. $M(S)$ is called the Grothendieck ring of finite type S -schemes. Notice that $M(S)$ depends only on the reduced structure of S .

2.2. (f^* and f_*). Given any morphism $f : T \rightarrow S$, there is functorial ring homomorphism $f^* : M(S) \rightarrow M(T)$ induced by base extension $X \rightarrow X \times_S T$. Moreover if f is itself of finite type, one also has a morphism of $M(S)$ -modules $f_* : M(T) \rightarrow M(S)$ induced by considering any T -scheme as an S -scheme via f .

Suppose we have a filtered inverse system of schemes $\{S_i\}_{i \in I}$ such that the inverse limit $\varprojlim S_i$ exists. Then one gets a natural ring homomorphism

$$\varinjlim M(S_i) \rightarrow M(\varprojlim S_i)$$

The following special case is of special interest.

2.3. Proposition. *Let S be an integral scheme. Let $\{U\}_{U \subset S}$ be the (filtered) inverse system of nonempty open sets of S . Let K be the function field of S .*

Then the natural ring homomorphism $\varinjlim_{U \subset S} M(U) \rightarrow M(\text{Spec}(K))$ is an isomorphism.

Proof. Any finite type K -scheme X_K , is the generic fibre of some finite type U -scheme X_U for some nonempty open set U of S . This shows the above map is surjective. After shrinking U if necessary, any closed subscheme $Z_K \subset X_K$ can be realized as the generic fibre of a closed subscheme $Z_U \subset X_U$. This shows the map is injective. \square

Finally, we recall the following elementary proposition.

2.4. Proposition (well-known). *Let \mathcal{E} be a vector bundle on S of rank $n + 1$. Then $[\text{Proj}(\mathcal{E})] = [\mathbb{P}_S^n]$ in $M(S)$.*

Proof. Let $S = \bigcup_{i=1}^m U_i$ be an open cover such that $\mathcal{E}|_{U_i}$ is trivial for each i . We now proceed by induction on m . If $m = 1$, \mathcal{E} is a trivial vector bundle and the statement is obvious. For $m > 1$, let $S' = \bigcup_{i=1}^{m-1} U_i$. Then by induction $[\text{Proj}(\mathcal{E}|_{S'})] = [\mathbb{P}_{S'}^n]$. Since \mathcal{E} is trivial on U_m and $S' \cap U_m$, we also have $[\text{Proj}(\mathcal{E}|_{U_m})] = [\mathbb{P}_{U_m}^n]$ and $[\text{Proj}(\mathcal{E}|_{S' \cap U_m})] = [\mathbb{P}_{S' \cap U_m}^n]$. The result now follows from the following equalities in $M(S)$.

$$\begin{aligned} [\text{Proj}(\mathcal{E})] &= [\text{Proj}(\mathcal{E}|_{S'})] + [\text{Proj}(\mathcal{E}|_{U_m})] - [\text{Proj}(\mathcal{E}|_{S' \cap U_m})] \\ [\mathbb{P}_S^n] &= [\mathbb{P}_{S'}^n] + [\mathbb{P}_{U_m}^n] - [\mathbb{P}_{S' \cap U_m}^n] \end{aligned}$$

\square

3. THE CREMONA MAP

3.1. (The Cremona Map) Let K be a field. Let us recall the following well known birational map from \mathbb{P}_K^2 to itself.

$$\phi : \mathbb{P}_K^2 \dashrightarrow \mathbb{P}_K^2 \quad [X, Y, Z] \rightarrow [YZ, XZ, XY]$$

ϕ can be defined everywhere on \mathbb{P}_K^2 outside the reduced closed subscheme

$$B = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$$

Let $X \xrightarrow{p} \mathbb{P}_K^2$ be the blow up of \mathbb{P}_K^2 with center B . Then ϕ defines a morphism $X \xrightarrow{q} \mathbb{P}_K^2$ such that the following diagram commutes.

$$\begin{array}{ccc} & X & \\ p \swarrow & & \searrow q \\ \mathbb{P}_K^2 & \xrightarrow{\phi} & \mathbb{P}_K^2 \end{array}$$

One can check that $q : X \rightarrow \mathbb{P}_K^2$ is itself is the blowup of \mathbb{P}_K^2 again with center B .

The following result was essentially proved in [4].

3.2. Theorem. *Let P and Q be Brauer Severi surfaces over a field K . Assume that $P = 2Q$ in $Br(K)$. Then there exists a birational map $\phi : P \dashrightarrow Q$ which after going to \overline{K} (the separable closure of K) is isomorphic to the Cremona map.*

3.3. Theorem. *Let K be any field and let P and Q be Brauer-Severi surfaces which generate the same subgroup in $Br(K)$. Then $[P] = [Q]$ in $M(K)$.*

Proof. Let $\phi : P \dashrightarrow Q$ be a map as guaranteed by Theorem(3.2). Let B (resp. B') be the base locus of the map ϕ (resp. ϕ^{-1}). Without loss of generality we may assume that P defines a nontrivial class in $Br(K)$ and thus has no K -point. Then B (resp. B') is a closed L -point (resp. L' -point) of P (resp. Q) for some degree 3 separable field extension L/K (resp. L'/K). We claim that L/K and L'/K are isomorphic field extensions. Let X be the blow up of P at B . Then we have the following Hironaka hut.

$$\begin{array}{ccc} & X & \\ p \swarrow & & \searrow q \\ P & \dashrightarrow \phi & Q \end{array}$$

Here q is the blowup of Q at B' (see (3.1)). To show that L/K and L'/K are isomorphic it is enough to show that $B' \times_K \text{Spec}(L)$ has an L -point. But after base extending to L , $P_L = P \times_K L \cong \mathbb{P}_L^2$ and p is the blow up of three L -points of P_L , say x, y, z . Let L_{xy} be the unique line in P_L joining x and y and similarly for L_{yz}, L_{xz} . Then the birational transform of $L_{xy} \cup L_{yz} \cup L_{xz}$ is the exceptional locus of q . Thus the image, $B' \times_K \text{Spec}(L)$, of this exceptional locus is the disjoint union of 3 points. This proves the claim.

Thus as K -varieties, the exceptional locus of p (resp. q) is isomorphic to \mathbb{P}_L^1 . Thus $[P] = [X] - [\mathbb{P}_L^1] + [\text{Spec}(L)] = [Q]$. \square

4. PROOF OF THE MAIN THEOREM

For any morphism $f : T \rightarrow S$ and any S -scheme X let $X_T = X \times_S T$.

4.1. Lemma. *Let X be any finite type S -scheme. Let P be a Brauer-Severi scheme over S of relative dimension n . Assume that the class of P in $Br(S)$ lies in the kernel of $Br(S) \rightarrow Br(X)$. Then*

$$[X \times P]_S = [\mathbb{P}_X^n]_S$$

Proof. It is enough to prove $[X \times P]_X = [\mathbb{P}_X^n]_X$ since the required equality can then be obtained by using the natural map $M(X) \rightarrow M(S)$. But by assumption, the class represented by $X \times P$ in $Br(X)$ is zero. Hence there exists a vector bundle \mathcal{E} on X such that $X \times P$ is isomorphic to $\text{Proj}(\mathcal{E})$ as X -schemes. The result now follows from (2.4). \square

Proof of (1.2). In order to avoid unnecessary repetition, we will only prove the theorem for Brauer-Severi surfaces. We proceed by induction on r .

(i) \Rightarrow (ii):

Step(1): One can quickly reduce to proving the statement in the case when S is integral. Let $S = S_1 \cup S_2$ be the decomposition of S into two closed subschemes. Then for any S -scheme X of finite type, we have

$$[X]_S = [X_{S_1}]_S + [X_{S_2}]_S - [X_{S_{12}}]_S \quad \text{where } S_{12} = S_1 \cap S_2$$

Hence by noetherian induction and the above formula, it is enough to prove the theorem in the case when S is irreducible. Moreover the natural ring homomorphism $M(S) \rightarrow M(S^{\text{red}})$ is an isomorphism. Thus we may assume S is integral.

Step(2):($r=1$) Let P/S and Q/S be two Brauer Severi surfaces. Let K be the function field of S . Then by (3.3)

$$[P_K] = [Q_K] \quad \text{in } M(K)$$

By (2.3), there exists a nonempty open set U of S such that

$$[P_U] = [Q_U] \quad \text{in } M(U)$$

Let Z be the complement of U . By noetherian induction $[P_Z] = [Q_Z]$. Since $[P] = [P_U] + [P_Z]$ and similarly for $[Q]$, we get that $[P]_S = [Q]_S$.

Step(3): Suppose the dimension of $\langle \{P_i\} \rangle$ (and hence also of $\langle \{Q_j\} \rangle$) as an \mathbb{F}_3 vector space is strictly less than r . Then without loss of generality we may assume that the class of P_r is contained in the subgroup generated by $\{P_i\}_{1 \leq i \leq r-1}$. Then by Corollary(4.1)

$$[\prod_i P_i] = [\prod_{i \leq r-1} P_i \times \mathbb{P}_S^2]$$

And similarly for the Q'_i s. By induction

$$[\prod_{1 \leq i \leq r-1} P_i] = [\prod_{1 \leq j \leq r-1} Q_j]$$

which implies

$$[\prod_{1 \leq i \leq r} P_i] = [\prod_{1 \leq i \leq r-1} P_i \times \mathbb{P}_S^2] = [\prod_{1 \leq j \leq r-1} Q_j \times \mathbb{P}_S^2] = [\prod_{1 \leq j \leq r} Q_j]$$

Hence it is enough to prove the theorem under the extra assumption that dimension of $\langle \{P_i\}_{1 \leq i \leq r} \rangle$ as an \mathbb{F}_3 vector space is r .

Step(4): Since class of Q_1 is in the subgroup generated by P'_i s in $Br(S)$, we have the following equation in $Br(S)$

$$Q_1 = \sum a_i P_i \quad , a_i \in \mathbb{F}_3$$

By *Step(3)*, at least one of the a'_i s is nonzero. Without loss of generality we may assume $a_1 \neq 0$. Thus $Q_1 = a_1 P_1 + \sum_{i \geq 2} a_i P_i$ in $Br(S)$. We first claim that

$$[Q_1 \times \prod_{2 \leq i \leq r} P_i]_S = [P_1 \times \prod_{2 \leq i \leq r} P_i]_S$$

Put $Y = \prod_{2 \leq i \leq r} P_i$. Now $Q_1 \times Y$ and $P_1 \times Y$ generate the same subgroup in $Br(Y)$. Hence by *Step(2)*, $[Q_1 \times Y]_Y = [P_1 \times Y]_Y$. The claim now follows by using the push forward map $M(Y) \rightarrow M(S)$. Now to prove the theorem it is enough to show

$$[Q_1 \times \prod_{2 \leq i \leq r} P_i]_S = [Q_1 \times \prod_{2 \leq i \leq r} Q_i]_S$$

But the subgroup generated by $\{P_i\}_{2 \leq i \leq r}$ in $Br(Q_1)$ is the same as the subgroup generated by $\{Q_i\}_{2 \leq i \leq r}$. Hence by induction on r we have

$$[Q_1 \times \prod_{2 \leq i \leq r} P_i]_{Q_1} = [Q_1 \times \prod_{2 \leq i \leq r} Q_i]_{Q_1}$$

Again, the claim follows by using the push forward map $M(Q_1) \rightarrow M(S)$.

(*i*) \Rightarrow (*iii*): (S is reduced): As in the proof of (*i*) \Rightarrow (*ii*), we proceed by induction on r . One first reduces the proof to the case when S is integral. Then by noetherian induction and the known result for the case when S is the spectrum of a field we prove case $r = 1$. After a possible re-indexing, one then proves $P_1 \times \prod_{2 \leq i \leq r} P_i$ is birational to $Q_1 \times \prod_{2 \leq i \leq r} Q_i$ by first comparing $P_1 \times \prod_{2 \leq i \leq r} P_i$ and $Q_1 \times \prod_{2 \leq i \leq r} P_i$ and then comparing $Q_1 \times \prod_{2 \leq i \leq r} P_i$ and $Q_1 \times \prod_{2 \leq i \leq r} Q_i$. Since the argument is very similar to the one above, we leave the details to the reader.

(S is a separated regular scheme): Without loss of generality we may assume S is connected. Let K be the function field of S . Since S is regular, $Br(S) \rightarrow Br(K)$ is injective. Thus in order to prove (*iii*) \Rightarrow (*i*) and (*ii*) \Rightarrow (*i*) we may replace S by $\text{Spec}(K)$. The kernel of $Br(K) \rightarrow Br(\prod P_i)$ is equal to $\langle \{P_i\} \rangle$. Moreover this kernel depends only on the stable birational class of $\prod P_i$. This proves that (*iii*) \Rightarrow (*i*).

If K is of characteristic zero then (*ii*) \Rightarrow (*i*) follows from the fact that any two smooth projective varieties having the same image in $M(K)$ are stably birational (see [3]). \square

Acknowledgement: I thank my advisor János Kollár for his encouragement and useful discussions. I also thank Chenyang Xu for useful comments and discussions.

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